Technical Appendix to
"Video Analysis of the Reentry of North Korea's July 28, 2017 Missile Test"

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This appendix describes the procedure used to reconstruct the three-dimensional trajectory of the object seen reentering Earth's atmosphere at the end of North Korea's July 28, 2017 missile test from videos of the event.

Definitions and physical model

The origin of the coordinate system (x, y, z) is the point on Earth's surface directly below the camera. The z-axis is aligned to the local vertical at that point. Within this coordinate system, the position of the camera is given by $\boldsymbol{x}_c = (0, 0, h_c)$. The camera is tilted at an angle α to the local horizontal (defined so that when $\alpha > 0$ the camera is pointing downwards). The x-axis is defined to be parallel to the camera's line of sight when $\alpha = 0$. The y-axis is perpendicular to both the x- and z-axes to form a normal right-handed set of axes.

We ignore atmospheric refraction and model the video image as the projection of the scene onto a flat "image plane," the normal vector of which is oriented along the camera's line of sight.[1] The perpendicular distance between this plane and the camera is denoted by a (note that the value of a depends on the screen on which the video is viewed). Because the normal vector to the image plane is $(-\cos\alpha, 0, \sin\alpha)$ and it passes through the point $\boldsymbol{p} = (a\cos\alpha, 0, h_c - a\sin\alpha)$, its equation can be shown to be

$$x\cos\alpha - z\sin\alpha = a - h_c\sin\alpha$$

Within the image plane, we define a second set of coordinates (y', z'), which is used in measuring the positions of points in the video image. The origin of this coordinate system is $\boldsymbol{p}$. The y'-axis is parallel to the y-axis. The z'-axis is perpendicular to both the image plane's normal and the y'-axis, that is, it is parallel to the vector $(\sin\alpha, 0, \cos\alpha)$. Note that when $\alpha = 0$, the z- and z'-axes are parallel.

With these definitions, the position in the (x, y, z) coordinate system of a point (y', z') on the image plane is given by

[1] We ultimately calculate the object in the video is more than 5 deg above the horizon. Formulae developed for astronomical observations (which presumably overestimate refraction when applied to an object within the atmosphere) suggest that refraction at this altitude is less than 0.2 deg. Refraction would, therefore, lead to an error of less than 3% in altitude estimation, which is probably smaller than other sources of error. The error due to refraction would decrease with increasing altitude. As a result, ignoring refraction slightly underestimates the speed of the object.

$$x = a\cos\alpha + z'\sin\alpha, \quad y = y', \quad \text{and} \quad z = h_c - a\sin\alpha + z'\cos\alpha \; (1)$$

Conversely, if the point (x, y, z) lies on the image plane, its position in the (y', z') coordinate system can be expressed as

$$y' = y, \text{ and } z' = x\sin\alpha + z\cos\alpha - h_c\cos\alpha \; (2)$$

Calibration

In order to reconstruct the video image in three dimensions, it is first necessary to estimate α, h_c, and a. Our approach starts from finding a set of points $\mathbf{x}_i = (x_i, y_i, z_i)$, the position of which in the physical world can be measured accurately. (For calibration purposes only, we take points close enough to the camera that the flat Earth approximation is applicable. The reconstruction procedure outlined below takes into account Earth's curvature.)

The line connecting point i to the camera is given by $\mathbf{r}_i = \mathbf{x}_c + \lambda_i (\mathbf{x}_i - \mathbf{x}_c)$, where λ_i is a real number. This line intersects the image plane when

$$\lambda_i = \frac{a}{x_i\cos\alpha - z_i\sin\alpha + h_c\sin\alpha}$$

Using (2), the coordinates (y_i', z_i') where point i should appear on the image plane are, therefore, given by

$$y'_i = \frac{a y_i}{x_i\cos\alpha - z_i\sin\alpha + h_c\sin\alpha}$$

and

$$z'_i = \frac{a(x_i\sin\alpha + z_i\cos\alpha - h_c\cos\alpha)}{x_i\cos\alpha - z_i\sin\alpha + h_c\sin\alpha}$$

In practice, because of atmospheric refraction, distortion caused by the camera's optics, measurement errors, and uncertainties in the exact values of x_i, y_i, and, in particular, z_i, the actual position on the image plane does not quite match the calculated position. Consequently, if (y_i', z_i') now denote the actual position, α, h_c, and a can be estimated numerically by finding the values that minimize the function

$$\sum_i \left(y'_i - \frac{a y_i}{x_i\cos\alpha - z_i\sin\alpha + h_c\sin\alpha}\right)^2 + \left(z'_i - \frac{a(x_i\sin\alpha + z_i\cos\alpha - h_c\cos\alpha)}{x_i\cos\alpha - z_i\sin\alpha + h_c\sin\alpha}\right)^2$$

Trajectory reconstruction

The goal is to reconstruct the trajectory of the object in the video from a series of measurements of its position on the screen, (y_i', z_i'), where the subscript i now indexes measurements at different times, t_i. The object is modelled to lie on the extension of the line connecting the camera to the position of its projection on the image plane. Using (1), its position $\mathbf{r}_i$ is given by

$$\boldsymbol{r}_i = (x_i, y_i, z_i) = \left(\lambda_i \cos\alpha + \lambda_i \frac{z'_i}{a}\sin\alpha, \lambda_i \frac{y'_i}{a}, h_c + \lambda_i \frac{z'_i}{a}\cos\alpha - \lambda_i \sin\alpha\right)$$

where λ_i is a real number to be determined. Note that y_i' and z_i' only appear as the ratios y_i'/a and z_i'/a. This ensures that the results for $\boldsymbol{r}_i$ do not depend on the screen on which the video is viewed (so long as *a* is correctly calculated for each screen). Conveniently, it also permits y_i', z_i' and *a* to be measured in pixels, but $\boldsymbol{r}_i$ to be calculated in more appropriate units (such as km).

To determine λ_i, we use the fact that the object land at a known distance, R_l, from the camera, measured along the surface of the Earth. Using the definition, $\theta_l = R_l/r_e$, where r_e is the radius of Earth, the position $\boldsymbol{r}_l$ where the object lands is given by

$$\boldsymbol{r}_l = r_e(\sin\theta_l \cos\varphi_l, \sin\theta_l \sin\varphi_l, \cos\theta_l - 1)$$

where φ_l, the angle between $\boldsymbol{r}_l$ and the *x*-axis. This angle can be estimated from the video, though is ultimately calculated more accurately.

To make further progress, we assume that the trajectory of the incoming object is linear, that is, the angle, θ_r, between its trajectory and $\boldsymbol{n}_l$ is constant, where $\boldsymbol{n}_l$ is the unit normal vector to Earth's surface at the point $\boldsymbol{r}_l$:

$$\boldsymbol{n}_l = (\sin\theta_l \cos\varphi_l, \sin\theta_l \sin\varphi_l, \cos\theta_l)$$

Especially for very lofted trajectories at the range of altitudes of interest, this is a good approximation across a wide range of ballistic coefficients. For example, computer modelling of an object with a ballistic coefficient of β = 500 lb/ft^2 re-entering Earth's atmosphere after travelling on the ballistic trajectory used by North Korea's July 28 test shows that an altitude of 60.3 km, θ_r = 6.2 deg. If θ_r were constant for the remainder of the flight, the missile would travel a further 6.6 km downrange. This value is very close to the figure of 6.5 km determined by the computer simulation. The approximation is not as good for a high drag re-entry object, but is still entirely adequate given the various other sources of error. Specifically, for an object with β = 50 lb/ft^2, θ_r is also 6.2 deg at an altitude of 60.3 km, again implying the object would travel a further 6.6 km downrange before impact. In this case, however, computer modelling suggests the correct value would be 5.3 km.

On the basis of this approximation scheme, the following procedure can then be used to reconstruct the trajectory of the object.

1. Estimate φ_l from the video.

2. Using this value, calculate $\boldsymbol{r}_l$ and $\boldsymbol{n}_l$.

3. For each measured position of the object on the video at time t_i, choose λ_i such that

$$\cos\theta_r = \frac{(\boldsymbol{r}_i - \boldsymbol{r}_l).\boldsymbol{n}_l}{|\boldsymbol{r}_i - \boldsymbol{r}_l|}$$

   Because solving this equation is geometrically equivalent to finding the point of intersection between a line and a cone, there are generally two solutions.[2] One of these solutions (where $\boldsymbol{r}_i$ is between the camera and $\boldsymbol{r}_l$) is unphysical and is discarded.

4. Using a linear regression on the set of points $\boldsymbol{r}_i$, calculate the point $\boldsymbol{r}^*_l$ at which the linearized trajectory impacts Earth's surface.

5. If the distance between $\boldsymbol{r}^*_l$ and $\boldsymbol{r}_l$ is less than some specified value (say, 10 m) then the results are self-consistent and the values $\boldsymbol{r}_i$ are kept. If the distance between $\boldsymbol{r}^*_l$ and $\boldsymbol{r}_l$ is greater than the specified value then the process is repeated from step 1, using an updated value of φ_l, calculated by finding the angle between $\boldsymbol{r}^*_l$ and the x-axis.

Having obtained $\boldsymbol{r}_i$ it is then straightforward to calculate the altitude and speed of the object along its trajectory.

[2] Or none in the case of a major snafu with the modelling.